

HIGHER HOMOTOPY HOPF ALGEBRA FOUND: A TEN YEAR RETROSPECTIVE

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ABSTRACT. The search for higher homotopy Hopf algebras (known today as A_∞ -bialgebras) began in 1996 during a conference at Vassar College honoring Jim Stasheff in the year of his 60th birthday. In a talk entitled "In Search of Higher Homotopy Hopf Algebras", I indicated that a DG Hopf algebra could be thought of as some (unknown) higher homotopy structure with trivial higher order structure and deformed using a graded version of Gerstenhaber and Schack's bialgebra deformation theory. In retrospect, the bi(co)module structure encoded in Gerstenhaber and Schack's differential defining deformation cohomology detects some (but not all) of the A_∞ -bialgebra structure relations. Nevertheless, this motivated the discovery of A_∞ -bialgebras by S. Sanedlidze and myself in 2005.

To Murray Gerstenhaber and Jim Stasheff

1. INTRODUCTION

In a preprint dated June 14, 2004, Samson Sanedlidze and I announced the definition of A_∞ -bialgebras [SU05], marking approximately six years of collaboration that continues to this day. Unknown to us at the time, A_∞ -bialgebras are ubiquitous and fundamentally important. Indeed, over a field F , the bialgebra structure on the singular chains of a loop space ΩX pulls back along a quasi-isomorphism $f : H_*(\Omega X; F) \rightarrow C_*(\Omega X)$ to an A_∞ -bialgebra structure on homology in a canonical way [SU08b].

Many have tried unsuccessfully to define A_∞ -bialgebras. The illusive ingredient in the definition turned out to be an explicit diagonal Δ_P on the permutahedra $P = \sqcup_{n \geq 1} P_n$, the first construction of which was given by S. Sanedlidze and myself in [SU04]. This paper is an account of the historical events leading up to the discovery of A_∞ -bialgebras and the truly remarkable role played by Δ_P in this regard. Although the ideas and examples presented here are quite simple, they represent and motivate general theory in [SU04], [SU05], [SU08a], and [SU08b].

Through their work in the theory of PROPs and the related area of infinity Lie bialgebras, many authors have contributed indirectly to this work, most notably M. Chas and D. Sullivan [CS04], V. Godin [God08], J-L. Loday [Lod06], M. Markl [Mar06], T. Pirashvili [Pir02], B. Shoikhet [Sho03], and B. Vallette [Val04]; for extensive bibliographies see [Sul07] and [Mar06].

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Several new results spin off of this discussion and are included here: Example 1 in Section 3 introduces the first example of a bialgebra H endowed with an A_∞ -algebra structure that is compatible with the comultiplication. Example 2 in Section 4, introduces the first example of a “non-operadic” A_∞ -bialgebra with a non-trivial operation $\omega^{2,2} : H^{\otimes 2} \rightarrow H^{\otimes 2}$. And in Section 5 we prove Theorem 1: *Given a DG bialgebra (H, d, μ, Δ) and a Gerstenhaber-Schack 2-cocycle $\mu_1^n \in \text{Hom}^{2-n}(H^{\otimes n}, H)$, $n \geq 3$, let $H_0 = (H[[t]], d, \mu, \Delta)$. Then $(H[[t]], d, \mu, \Delta, t\mu_1^n)$ is a linear deformation of H_0 as a simple Hopf $A(n)$ -algebra.*

2. THE HISTORICAL CONTEXT

Two papers with far-reaching consequences in algebra and topology appeared in 1963. In [Ger63] Murray Gerstenhaber introduced the deformation theory of associative algebras and in [Sta63] Jim Stasheff introduced the notion of an $A(n)$ -algebra. Although the notion of what we now call a “non- Σ operad” appears in both papers, this connection went unnoticed until after Jim’s visit to the University of Pennsylvania in 1983. Today, Gerstenhaber’s deformation theory and Stasheff’s higher homotopy algebras are fundamental tools in algebra, topology and physics. An extensive bibliography of applications appears in [MSS02].

By 1990, techniques from deformation theory and higher homotopy structures had been applied by many authors, myself included [Umb89], [LU92], to classify rational homotopy types with a fixed cohomology algebra. And it seemed reasonable to expect that rational homotopy types with a fixed Pontryagin algebra $H_*(\Omega X; \mathbb{Q})$ could be classified in a similar way. Presumably, such a theory would involve deformations of DG bialgebras (DGBs) as some higher homotopy structure with compatible A_∞ -algebra and A_∞ -coalgebra substructures, but the notion of compatibility was not immediately clear and an appropriate line of attack seemed illusive. But one thing was clear: If we apply a graded version of Gerstenhaber and Schack’s (G-S) deformation theory [GS92], [LM91], [LM96], [Umb96] and deform a DGB H as some (unknown) higher homotopy structure, new operations $\omega^{j,i} : H^{\otimes i} \rightarrow H^{\otimes j}$ appear and their interactions with the deformed bialgebra operations are partially detected by the differentials. While this is but one small piece of a very large puzzle, it gave us a clue.

During the conference honoring Jim Stasheff in the year of his 60th birthday, held at Vassar College in June 1996, I discussed this particular clue in a talk entitled “*In Search of Higher Homotopy Hopf Algebras*” ([McC98] p. xii). Although G-S deformations of DGBs are less constrained than the A_∞ -bialgebras known today, they motivated the definition announced eight years later.

Following the Vassar conference, forward progress halted. Questions of structural compatibility seemed mysterious and inaccessible. Then in 1998, Jim Stasheff ran across some related work by S. Sanedlidze [San96], of the A. Razmadze Mathematical Institute in Tbilisi, and suggested that I get in touch with him. Thus began our long and fruitful collaboration. Over the months that followed, Sanedlidze applied techniques of homological perturbation theory to solve the aforementioned classification problem [San99], but the higher order structure in the limit is implicit and the structure relations are inaccessible. In retrospect, this is not surprising as explicit structure relations require explicit combinatorial diagonals Δ_P on the permutahedra $P = \sqcup_{n \geq 1} P_n$ and Δ_K on the associahedra $K = \sqcup_{n \geq 2} K_n$. But such diagonals are difficult to construct and were unknown to us at the time. Indeed,

one defines the tensor product of A_∞ -algebras in terms of Δ_K , and the search for a construction of Δ_K had remained a long-standing problem in the theory of operads. We announced our construction of Δ_K in 2000 [SU00]; our construction of Δ_P followed a year or two later (see [SU04]).

3. TWO IMPORTANT ROLES FOR Δ_P

The diagonal Δ_P plays two fundamentally important roles in the theory of A_∞ -bialgebras. First, one builds the structure relations from components of free extensions of initial maps as higher (co)derivations with respect to Δ_P , and second, Δ_P specifies exactly which of these components to use.

To appreciate the first of these roles, recall the following definition given by Stasheff in his seminal work on A_∞ -algebras in 1963 [Sta63]: Let A be a graded module, let $\{\mu^i \in \text{Hom}^{i-2}(A^{\otimes i}, A)\}_{i \geq 1}$ be an arbitrary family of maps, and let d be the cofree extension of $\Sigma \mu^i$ as a coderivation of the tensor coalgebra $T^c A$ (with a shift in dimension). Then (A, μ^i) is an A_∞ -algebra if $d^2 = 0$; when this occurs, the universal complex $(T^c A, d)$ is called the *tilde-bar construction* and the structure relations in A are the homogeneous components of $d^2 = 0$. Similarly, let H be a graded module and let $\{\omega^{j,i} \in \text{Hom}^{3-i-j}(H^{\otimes i}, H^{\otimes j})\}_{i,j \geq 1}$ be an arbitrary family of maps. When $(H, \omega^{j,i})$ is an A_∞ -bialgebra, there is an associated universal complex $(Bd(H), \bar{\omega})$, called the *biderivative* of $\omega = \Sigma \omega^{j,i}$ (constructed by Saneblidze and myself in [SU05]) whose differential $\bar{\omega}$ is the sum of various (co)free extensions of various subfamilies of $\{\omega^{j,i}\}$ as Δ_P -(co)derivations. And indeed, the structure relations in H are the homogeneous components of $\bar{\omega}^2 = 0$.

To demonstrate the spirit of this, consider a free graded module H of finite type and an (arbitrary) map $\omega = \mu + \mu^3 + \Delta$ with components $\mu : H^{\otimes 2} \rightarrow H$, $\mu^3 : H^{\otimes 3} \rightarrow H$, and $\Delta : H \rightarrow H^{\otimes 2}$. Extend Δ as an algebra map $\bar{\Delta} : T^a H \rightarrow T^a(H^{\otimes 2})$, extend $\mu + \mu^3$ as a coderivation $d : T^c H \rightarrow T^c H$, and extend μ as a coalgebra map $\bar{\mu} : T^c(H^{\otimes 2}) \rightarrow T^c H$. Finally, note that $f = (\mu \otimes 1)\mu$ and $g = (1 \otimes \mu)\mu$ are coalgebra maps, and extend μ^3 as an (f, g) -coderivation $\bar{\mu}^3 : T^c(H^{\otimes 3}) \rightarrow T^c H$. Then

$$\bar{\omega} = d + \bar{\mu} + \bar{\mu}^3 + \bar{\Delta} \in \bigoplus_{p,q,r,s \geq 1} \text{Hom}\left((H^{\otimes p})^{\otimes q}, (H^{\otimes r})^{\otimes s}\right).$$

Let $\sigma_{r,s} : (H^{\otimes r})^{\otimes s} \rightarrow (H^{\otimes s})^{\otimes r}$ denote the canonical permutation of tensor factors and define a composition product \odot for homogeneous components A and B of $\bar{\omega} = d + \bar{\mu} + \bar{\mu}^3 + \bar{\Delta}$ by

$$A \odot B = \begin{cases} A \circ \sigma_{r,s} \circ B, & \text{if defined} \\ 0, & \text{otherwise;} \end{cases}$$

when $A \odot B$ is defined, $(H^{\otimes r})^{\otimes s}$ is the target of B , and $(H^{\otimes s})^{\otimes r}$ is the source of A . Then (H, μ, μ^3, Δ) is an A_∞ -infinity bialgebra if $\bar{\omega} \odot \bar{\omega} = 0$. Note that $\Delta\mu$ and $(\mu \otimes \mu) \sigma_{2,2}(\Delta \otimes \Delta)$ are the homogeneous components of $\bar{\omega} \odot \bar{\omega}$ in $\text{Hom}(H^{\otimes 2}, H^{\otimes 2})$; consequently, $\bar{\omega} \odot \bar{\omega} = 0$ implies the Hopf relation

$$\Delta\mu = (\mu \otimes \mu) \sigma_{2,2}(\Delta \otimes \Delta).$$

Now if (H, μ, Δ) is a bialgebra, the operations μ_t , μ_t^3 , and Δ_t in a G-S deformation of H satisfy

$$(3.1) \quad \Delta_t \mu_t^3 = [\mu_t(\mu_t \otimes 1) \otimes \mu_t^3 + \mu_t^3 \otimes \mu_t(1 \otimes \mu_t)] \sigma_{2,3} \Delta_t^{\otimes 3}$$

and the homogeneous components of $\overline{\omega} \odot \overline{\omega} = 0$ in $Hom(H^{\otimes 3}, H^{\otimes 2})$ are exactly those in (3.1). So this is encouraging.

Recall that the permutahedron P_1 is a point 0 and P_2 is an interval 01. In these cases Δ_P agrees with the Alexander-Whitney diagonal on the simplex:

$$\Delta_P(0) = 0 \otimes 0 \quad \text{and} \quad \Delta_P(01) = 0 \otimes 01 + 01 \otimes 1.$$

If X is an n -dimensional cellular complex, let $C_*(X)$ denote the cellular chains of X . When X has a single top dimensional cell, we denote it by e^n . An A_∞ -algebra structure $\{\mu^n\}_{n \geq 2}$ on H is encoded operadically by a family of chain maps

$$\{\xi : C_*(P_{n-1}) \rightarrow Hom(H^{\otimes n}, H)\},$$

which factor through the map $\theta : C_*(P_{n-1}) \rightarrow C_*(K_n)$ induced by cellular projection $P_{n-1} \rightarrow K_n$ given by A. Tonks [Ton97] and satisfy $\xi(e^{n-2}) = \mu^n$. The fact that

$$(\xi \otimes \xi) \Delta_P(e^0) = \mu \otimes \mu \quad \text{and}$$

$$(\xi \otimes \xi) \Delta_P(e^1) = \mu(\mu \otimes 1) \otimes \mu^3 + \mu^3 \otimes \mu(1 \otimes \mu)$$

are components of $\overline{\mu}$ and $\overline{\mu}^3$ suggests that we extend a given μ^n as a higher coderivation $\overline{\mu}^n : T^c(H^{\otimes n}) \rightarrow T^c H$ with respect to Δ_P . Indeed, an A_∞ -bialgebra of the form $(H, \Delta, \mu^n)_{n \geq 2}$ is defined in terms of the usual A_∞ -algebra relations together with the relations

$$(3.2) \quad \Delta \mu^n = [(\xi \otimes \xi) \Delta_P(e^{n-2})] \sigma_{2,n} \Delta^{\otimes n},$$

which define the compatibility of μ^n and Δ .

Structure relations in more general A_∞ -bialgebras of the form $(H, \Delta^n, \mu^n)_{n \geq 2}$ are similar in spirit and formulated in [Umb08]. Special cases of the form $(H, \Delta, \Delta^n, \mu)$ with a single Δ^n were studied by H.J. Baues in the case $n = 3$ [Bau98] and by A. Berciano and myself in cases $n \geq 3$ [BU08]. Indeed, if p is an odd prime and $n \geq 3$, these particular structures appear as tensor factors of the mod p homology of an Eilenberg-Mac Lane space of type $K(\mathbb{Z}, n)$.

Dually, A_∞ -bialgebras (H, Δ, μ, μ^n) with a single μ^n have a strictly associative multiplication μ and $\xi \otimes \xi$ acts exclusively on the primitive terms of Δ_P for lacunary reasons, in which case relation (3.2) reduces to

$$(3.3) \quad \Delta \mu^n = (f_n \otimes \mu^n + \mu^n \otimes f_n) \sigma_{2,n} \Delta^{\otimes n},$$

where $f_n = \mu(\mu \otimes 1) \cdots (\mu \otimes 1^{\otimes n-2})$. The first example of this particular structure now follows.

Example 1. Let H be the primitively generated bialgebra $\Lambda(x, y)$ with $|x| = 1, |y| = 2$, and

$$\mu^n(x^{i_1} y^{p_1} | \cdots | x^{i_n} y^{p_n}) = \begin{cases} y^{p_1 + \cdots + p_n + 1}, & i_1 \cdots i_n = 1 \text{ and } p_k \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

One can easily check that H is an A_∞ -algebra, and a straightforward calculation together with the identity

$$\binom{p_1 + \cdots + p_n + 1}{i} = \sum_{s_1 + \cdots + s_n = i-1} \binom{p_1}{s_1} \cdots \binom{p_n}{s_n} + \sum_{s_1 + \cdots + s_n = i} \binom{p_1}{s_1} \cdots \binom{p_n}{s_n}$$

verifies relation (3.3).

The second important role played by Δ_P is evident in A_∞ -bialgebras in which $\omega^{n,m}$ is non-trivial for some $m, n > 1$. Just as an A_∞ -algebra structure on H is encoded operadically, an A_∞ -bialgebra structure on H is encoded *matradically* by a family of chain maps

$$\{\varepsilon : C_*(KK_{n,m}) \rightarrow \text{Hom}^{3-m-n}(H^{\otimes m}, H^{\otimes n})\}$$

over contractible polytopes $KK = \sqcup_{m,n \geq 1} KK_{n,m}$, called *matrahedra*, with single top dimensional cells e^{m+n-3} such that $\varepsilon(e^{m+n-3}) = \omega^{n,m}$. Note that $KK_{n,1} = KK_{1,n}$ is the associahedron K_n [SU08a]. Let $M = \{M_{n,m} = \text{Hom}(H^{\otimes m}, H^{\otimes n})\}$ and let $\Theta = \{\theta_m^n = \omega^{n,m}\}$. The A_∞ -bialgebra matrad \mathcal{H}_∞ is realized by $C_*(KK)$ and is a proper submodule of the free PROP generated by Θ . The *matrad product* γ on \mathcal{H}_∞ is defined in terms of Δ_P , and a monomial α in the free PROP is a component of a structure relation if and only if $\alpha \in \mathcal{H}_\infty$.

More precisely, in [Mar06] M. Markl defined the submodule S of *special elements* in PROP whose additive generators are monomials α expressed as “elementary fractions”

$$(3.4) \quad \alpha = \frac{\alpha_p^{y_1} \cdots \alpha_p^{y_q}}{\alpha_{x_1}^q \cdots \alpha_{x_p}^q}$$

in which $\alpha_{x_i}^q$ and $\alpha_p^{y_j}$ are additive generators of S and the j^{th} output of $\alpha_{x_i}^q$ is linked to the i^{th} input of $\alpha_p^{y_j}$ (here juxtaposition denotes tensor product). Representing θ_m^n graphically as a double corolla (see Figure 1), a general decomposable α is represented by a connected non-planar graph in which the generators appear in order from left-to-right (see Figure 2). The matrad \mathcal{H}_∞ is a proper submodule of S and the matrad product γ agrees with the restriction of Markl’s fraction product to \mathcal{H}_∞ .

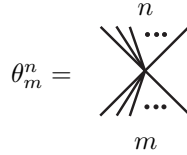


Figure 1.

The diagonal Δ_P acts as a filter and admits certain elementary fractions as additive generators of \mathcal{H}_∞ . In dimensions 0 and 1, the diagonal Δ_P is expressed graphically in terms of up-rooted planar rooted trees (with levels) by

$$\Delta_P(\text{Y}) = \text{Y} \otimes \text{Y} \quad \text{and} \quad \Delta_P(\text{Y}) = \text{Y} \otimes \text{Y} + \text{Y} \otimes \text{Y}.$$

Define $\Delta_P^{(0)} = 1$; for each $k \geq 1$, define $\Delta_P^{(k)} = (\Delta_P \otimes 1^{\otimes k-1}) \Delta_P^{(k-1)}$ and view each component of $\Delta_P^{(k)}(\theta_q^1)$ as a $(q-2)$ -dimensional subcomplex of $(P_{q-1})^{\times k+1}$, and similarly for $\Delta_P^{(k)}(\theta_1^q)$.

The elements θ_1^1 , θ_2^1 , and θ_1^2 generate two elementary fractions in $M_{2,2}$ each of dimension zero, namely,

$$\alpha_2^2 = \frac{\text{Y}}{\text{Y}} \quad \text{and} \quad \alpha_{11}^{11} = \frac{\text{Y} \otimes \text{Y}}{\text{Y} \otimes \text{Y}}.$$

Define $\partial(\theta_2^2) = \alpha_2^2 + \alpha_{11}^{11}$, and label the edge and vertices of the interval $KK_{2,2}$ by θ_2^2 , α_2^2 and α_{11}^{11} , respectively. Continuing inductively, the elements θ_1^1 , θ_2^1 , θ_1^2 , θ_2^2 , α_2^2 , and α_{11}^{11} generate 18 fractions in $M_{2,3}$ – one in dimension 2, nine in dimension 1 and eight in dimension 0. Of these, 14 label the edges and vertices of the heptagon $KK_{2,3}$. Since the generator θ_3^2 must label the 2-face, we wish to discard the 2-dimensional decomposable

$$e = \frac{\begin{array}{c} \diagup \quad \diagup \\ \diagdown \quad \diagdown \end{array}}{\begin{array}{c} \diagdown \quad \diagdown \\ \diagup \quad \diagup \end{array}}$$

and the appropriate components of its boundary. Note that e is a square whose boundary is the union of four edges

$$(3.5) \quad \frac{\begin{array}{c} \diagup \quad \diagup \\ \diagdown \quad \diagdown \end{array}}{\begin{array}{c} \diagdown \quad \diagdown \\ \diagup \quad \diagup \end{array}}, \quad \frac{\begin{array}{c} \diagup \quad \diagup \\ \diagdown \quad \diagdown \end{array}}{\begin{array}{c} \diagdown \quad \diagdown \\ \diagup \quad \diagup \end{array}}, \quad \frac{\begin{array}{c} \diagup \quad \diagup \\ \diagdown \quad \diagdown \end{array}}{\begin{array}{c} \diagdown \quad \diagdown \\ \diagup \quad \diagup \end{array}}, \text{ and } \frac{\begin{array}{c} \diagup \quad \diagup \\ \diagdown \quad \diagdown \end{array}}{\begin{array}{c} \diagdown \quad \diagdown \\ \diagup \quad \diagup \end{array}}.$$

Of the five fractions pictured above, only the first two in (3.5) have numerators and denominators that are components of $\Delta_P^{(k)}(P)$ (numerators are components of $\Delta_P^{(1)}(\theta_3^1)$ and denominators are exactly $\Delta_P^{(2)}(\theta_1^2)$). Our selection rule admits only these two particular fractions, leaving seven 1-dimensional generators to label the edges of $KK_{2,3}$ (see Figure 2). Now linearly extend the boundary map ∂ to the seven admissible 1-dimensional generators and compute the seven 0-dimensional generators labeling the vertices of $KK_{2,3}$. Since the 0-dimensional generator

$$\frac{\begin{array}{c} \diagup \quad \diagup \\ \diagdown \quad \diagdown \end{array}}{\begin{array}{c} \diagdown \quad \diagdown \\ \diagup \quad \diagup \end{array}}$$

is not among them, we discard it.

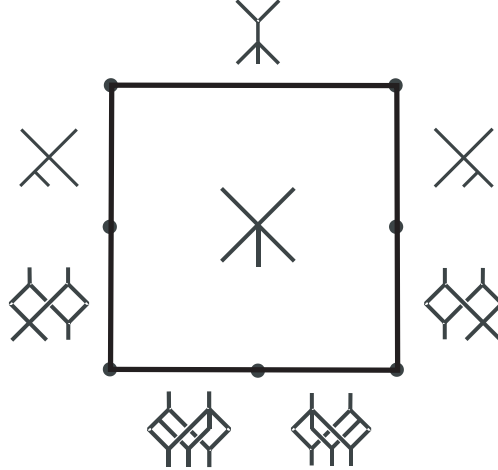


Figure 2: The matrahedron $KK_{2,3}$.

Subtleties notwithstanding, this process continues indefinitely and produces \mathcal{H}_∞ as an explicit free resolution of the bialgebra matrad $\mathcal{H} = \langle \theta_1^1, \theta_2^1, \theta_1^2 \rangle$ in the category of matrads. We note that in [Mar06], M. Markl makes arbitrary choices (independent of our selection rule) to construct the polytopes $B_m^n = KK_{n,m}$ for

$m + n \leq 6$. In this range, it is enough to consider components of the diagonal Δ_K on the associahedra.

We conclude this section with a brief review of our diagonals Δ_P and Δ_K (up to sign); for details see [SU04]. Alternative constructions of Δ_K were subsequently given by Markl and Shnider [MS06] and J.L. Loday [Lod07] (in this volume). Let $\underline{n} = \{1, 2, \dots, n\}$, $n \geq 1$. A matrix E with entries from $\{0\} \cup \underline{n}$ is a *step matrix* if:

- Each element of \underline{n} appears as an entry of E exactly once.
- The elements of \underline{n} in each row and column of E form an increasing contiguous block.
- Each diagonal parallel to the main diagonal of E contains exactly one element of \underline{n} .

Right-shift and down-shift matrix transformations, which include the identity (a trivial shift), act on step matrices and produce *derived matrices*. Let $a = A_1|A_2|\dots|A_p$ and $b = B_q|B_{q-1}|\dots|B_1$ be partitions of \underline{n} . The pair $a \times b$ is an (p, q) -complementary pair (CP) if B_i and A_j are the rows and columns of a $q \times p$ derived matrix. Since faces of P_n are indexed by partitions of \underline{n} , and CPs are in one-to-one correspondence with derived matrices, each CP is identified with some product face of $P_n \times P_n$.

Definition 1. Define $\Delta_P(e^0) = e^0 \otimes e^0$. Inductively, having defined Δ_P on $C_*(P_{k+1})$ for all $0 \leq k \leq n-1$, define Δ_P on $C_n(P_{n+1})$ by

$$\Delta_P(e^n) = \sum_{\substack{(p,q)\text{-CPs } u \times v \\ p+q=n+2}} \pm u \otimes v,$$

and extend multiplicatively to all of $C_*(P_{n+1})$.

The diagonal Δ_P induces a diagonal Δ_K on $C_*(K)$. Recall that faces of P_n in codimension k are indexed by planar rooted trees with $n+1$ leaves and $k+1$ levels (PLTs), and forgetting levels defines the cellular projection $\theta : P_n \rightarrow K_{n+1}$ given by A. Tonks [Ton97]. Thus faces of P_n indexed by PLTs with multiple nodes in the same level degenerate under θ , and corresponding generators lie in the kernel of the induced map $\theta : C_*(P_n) \rightarrow C_*(K_{n+1})$. The diagonal Δ_K is given by $\Delta_K \theta = (\theta \otimes \theta) \Delta_P$.

4. DEFORMATIONS OF DG BIALGEBRAS AS $A(n)$ -BIALGEBRAS

The discussion above provides the context to appreciate the extent to which G-S deformation theory motivates the notion of an A_∞ -bialgebra. We describe this motivation in this section. In retrospect, the bi(co)module structure encoded in the G-S differentials controls some (but not all) of the A_∞ -bialgebra structure relations. For example, all structure relation in A_∞ -bialgebras of the form $(H, d, \mu, \Delta, \mu^n)$ are controlled except

$$(4.1) \quad \sum_{i=0}^{n-1} (-1)^{i(n+1)} \mu^n (1^{\otimes i} \otimes \mu^n \otimes 1^{\otimes n-i-1}) = 0,$$

which measures the interaction of μ^n with itself. Nevertheless, such structures admit an $A(n)$ -algebra substructure and their single higher order operation μ^n is compatible with Δ . Thus we refer to such structures here as *Hopf $A(n)$ -algebras*.

General G-S deformations of DGBs, referred to here as *quasi- $A(n)$ -bialgebras*, are “partial” $A(n)$ -bialgebras in the sense that all structure relations involving multiple higher order operations are out of control.

4.1. $A(n)$ -Algebras and Their Duals. The signs in the following definition were given in [SU04] and differ from those given by Stasheff in [Sta63]. We note that either choice of signs induces an oriented combinatorial structure on the associahedra, and these structures are equivalent. Let $n \in \mathbb{N} \cup \{\infty\}$.

Definition 2. An $A(n)$ -algebra is a graded module A together with structure maps $\{\mu^k \in \text{Hom}^{2-k}(A^{\otimes k}, A)\}_{1 \leq k < n+1}$ that satisfy the relations

$$\sum_{j=0}^{k-1} \sum_{i=0}^{k-j-1} (-1)^{j(i+1)} \mu^{k-j} (1^{\otimes i} \otimes \mu^{j+1} \otimes 1^{\otimes k-j-1-i}) = 0$$

for each $k < n+1$. Dually, an $A(n)$ -coalgebra is a graded module C together with structure maps $\{\Delta^k \in \text{Hom}^{2-k}(C, C^{\otimes k})\}_{1 \leq k < n+1}$ that satisfy the relations

$$\sum_{j=0}^{k-1} \sum_{i=0}^{k-j-1} (-1)^{j(k+i+1)} (1^{\otimes i} \otimes \Delta^{j+1} \otimes 1^{\otimes k-j-1-i}) \Delta^{k-j} = 0$$

for each $k < n+1$.

An $A(n)$ -algebra is *strict* if $n < \infty$ and $\mu^n = 0$. A *simple* $A(n)$ -algebra is a strict $A(n+1)$ -algebra of the form (A, d, μ, μ^n) ; in particular, a simple $A(3)$ -algebra is a strict $A(4)$ -algebra in which

- i. d is both a differential and a derivation of μ ,
- ii. μ is homotopy associative and μ^3 is an associating homotopy:

$$d\mu^3 + \mu^3(d \otimes 1 \otimes 1 + 1 \otimes d \otimes 1 + 1 \otimes 1 \otimes d) = \mu(\mu \otimes 1) - \mu(1 \otimes \mu),$$

- iii. μ and μ^3 satisfy a strict pentagon condition:

$$\mu^3(\mu \otimes 1 \otimes 1 - 1 \otimes \mu \otimes 1 + 1 \otimes 1 \otimes \mu) = \mu(1 \otimes \mu^3 + \mu^3 \otimes 1).$$

4.2. Deformations of DG Bialgebras. In [GS92], M. Gerstenhaber and S. D. Schack defined the cohomology of an *ungraded* bialgebra by joining the dual cohomology theories of G. Hochschild [HKR62] and P. Cartier [Car55]. This construction was given independently by A. Lazarev and M. Movshev in [LM91]. The *G-S cohomology of H* reviewed here is a straight-forward extension to the graded case and was constructed in [LM96] and [Umb96].

Let (H, d, μ, Δ) be a connected DGB. We assume $|d| = 1$, although one could assume $|d| = -1$ equally well. For detailed derivations of the formulas that follow see [Umb96]. For each $i \geq 1$, the *i -fold bicomodule tensor power of H* is the H -bicomodule $H^{\otimes i} = (H^{\otimes i}, \lambda_i, \rho_i)$ with left and right coactions given by

$$\lambda_i = [\mu(\mu \otimes 1) \cdots (\mu \otimes 1^{\otimes i-2}) \otimes 1^{\otimes i}] \sigma_{2,i} \Delta^{\otimes i} \quad \text{and}$$

$$\rho_i = [1^{\otimes i} \otimes \mu(1 \otimes \mu) \cdots (1^{\otimes i-2} \otimes \mu)] \sigma_{2,i} \Delta^{\otimes i}.$$

When $f : H^{\otimes i} \rightarrow H^{\otimes*}$, there is the composition

$$(1 \otimes f) \lambda_i = [\mu(\mu \otimes 1) \cdots (\mu \otimes 1^{\otimes i-2}) \otimes f] \sigma_{2,i} \Delta^{\otimes i}.$$

Dually, for each $j \geq 1$, the *j -fold bimodule tensor power of H* is the H -bimodule $H^{\bar{\otimes} j} = (H^{\otimes j}, \lambda^j, \rho^j)$ with left and right actions given by

$$\lambda^j = \mu^{\otimes j} \sigma_{j,2} [(\Delta \otimes 1^{\otimes j-2}) \cdots (\Delta \otimes 1) \Delta \otimes 1^{\otimes j}] \text{ and}$$

$$\rho^j = \mu^{\otimes j} \sigma_{j,2} [1^{\otimes j} \otimes (1^{\otimes j-2} \otimes \Delta) \cdots (1 \otimes \Delta) \Delta] .$$

When $g : H^{\otimes*} \rightarrow H^{\otimes j}$, there is the composition

$$\lambda^j (1 \otimes g) = \mu^{\otimes j} \sigma_{j,2} [(\Delta \otimes 1^{\otimes j-2}) \cdots (\Delta \otimes 1) \Delta \otimes g] .$$

Let \mathbf{k} be a field. Extend d, μ and Δ to $\mathbf{k}[[t]]$ -linear maps and obtain a $\mathbf{k}[[t]]$ -DGB $H_0 = (H[[t]], d, \mu, \Delta)$. We wish to deform H_0 as an $A(n)$ -structure of the form

$$H_t = \left(H[[t]], d_t = \omega_t^{1,1}, \mu_t = \omega_t^{1,2}, \Delta_t = \omega_t^{2,1}, \omega_t^{j,i} \right)_{i+j=n+1} \text{ where}$$

$$\omega_t^{j,i} = \sum_{k=0}^{\infty} t^k \omega_k^{j,i} \in \text{Hom}^{3-i-j} (H^{\otimes i}, H^{\otimes j}),$$

$$\omega_0^{1,1} = d, \omega_0^{1,2} = \mu, \omega_0^{2,1} = \Delta, \text{ and } \omega_0^{j,i} = 0.$$

Deformations of H_0 are controlled by the G - S n -complex, which we now review. For $k \geq 1$, let

$$d_{(k)} = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes d \otimes 1^{\otimes k-i-1}$$

$$\partial_{(k)} = \sum_{i=0}^{k-1} (-1)^i 1^{\otimes i} \otimes \mu \otimes 1^{\otimes k-i-1}$$

$$\delta_{(k)} = \sum_{i=0}^{k-1} (-1)^i 1^{\otimes i} \otimes \Delta \otimes 1^{\otimes k-i-1}.$$

These differentials induce strictly commuting differentials d, ∂ , and δ on the tri-graded module $\{\text{Hom}^p(H^{\otimes i}, H^{\otimes j})\}$, which act on an element f in tridegree (p, i, j) by

$$d(f) = d_{(j)} f - (-1)^p f d_{(i)}$$

$$\partial(f) = \lambda^j (1 \otimes f) - f \partial_{(i)} - (-1)^i \rho^j (f \otimes 1)$$

$$\delta(f) = (1 \otimes f) \lambda_i - \delta_{(j)} f - (-1)^j (f \otimes 1) \rho_i.$$

The submodule of *total G - S r -cochains on H* is

$$C_{GS}^r(H, H) = \bigoplus_{p+i+j=r+1} \text{Hom}^p(H^{\otimes i}, H^{\otimes j})$$

and the total differential D on a cochain f in tridegree (p, i, j) is given by

$$D(f) = \left[(-1)^{i+j} d + \partial + (-1)^i \delta \right] (f),$$

where the sign coefficients are chosen so that (1) $D^2 = 0$, (2) structure relations (ii) and (iii) in Definition 2 hold and (3) the restriction of D to the submodule of r -cochains in degree $p = 0$ agrees with the total (ungraded) G - S differential. The G - S cohomology of H with coefficients in H is given by

$$H_{GS}^*(H, H) = H_* \{C_{GS}^r(H, H), D\}.$$

Identify $\text{Hom}^p(H^{\otimes i}, H^{\otimes j})$ with the point (p, i, j) in \mathbb{R}^3 . Then the G - S n -complex is that portion of the G - S complex in the region $x \geq 2 - n$ and the submodule of total r -cochains in the n -complex is

$$C_{GS}^r(H, H; n) = \bigoplus_{p=r-i-j+1 \geq 2-n} \text{Hom}^p(H^{\otimes i}, H^{\otimes j})$$

(a 2-cocycle in the 3-complex appears in Figure 3). The G - S n -cohomology of H with coefficients in H is given by

$$H_{GS}^*(H, H; n) = H_* \{C_{GS}^r(H, H; n); D\}.$$

Note that a general 2-cocycle α has a component of tridegree $(3 - i - j, i, j)$ for each i and j in the range $2 \leq i + j \leq n + 1$. Thus α has $n(n + 1)/2$ components and a standard result in deformation theory tells us that the homogeneous components of α determine an *infinitesimal deformation*, i.e., the component $\omega_1^{j,i}$ in tridegree $(3 - i - j, i, j)$ defines the first order approximation $\omega_0^{j,i} + t\omega_1^{j,i}$ of the structure map $\omega_t^{j,i}$ in H_t .

For simplicity, consider the case $n = 3$. Each of the ten homogeneous components of the deformation equation $D(\alpha) = 0$ produces the infinitesimal form of one structure relation (see below). In particular, a deformation H_t with structure maps $\{\omega_t^{1,i}\}_{1 \leq i \leq 3}$ is a simple $A(3)$ -algebra and a deformation H_t with structure maps $\{\omega_t^{j,1}\}_{1 \leq j \leq 3}$ is a simple $A(3)$ -coalgebra.

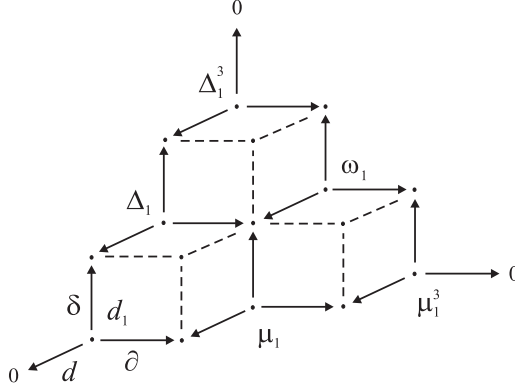


Figure 3. The 2-cocycle $d_1 + \mu_1 + \Delta_1 + \mu_1^3 + \omega_1 + \Delta_1^3$.

For notational simplicity, let $\mu_t^3 = \omega_t^{1,3}$, $\omega_t = \omega_t^{2,2}$ and $\Delta_t^3 = \omega_t^{3,1}$, and consider a deformation of (H, d, μ, Δ) as a “quasi- $A(3)$ -structure.” Then

- $d_t = d + td_1 + t^2d_2 + \cdots$
- $\mu_t = \mu + t\mu_1 + t^2\mu_2 + \cdots$
- $\Delta_t = \Delta + t\Delta_1 + t^2\Delta_2 + \cdots$
- $\mu_t^3 = t\mu_1^3 + t^2\mu_2^3 + \cdots$
- $\omega_t = t\omega_1 + t^2\omega_2 + \cdots$
- $\Delta_t^3 = t\Delta_1^3 + t^2\Delta_2^3 + \cdots$

and $d_1 + \mu_1 + \Delta_1 + \mu_1^3 + \omega_1 + \Delta_1^3$ is a total 2-cocycle (see Figure 3). Equating coefficients in $D(d_1 + \mu_1 + \Delta_1 + \mu_1^3 + \omega_1 + \Delta_1^3) = 0$ gives

1. $d(d_1) = 0$
2. $d(\mu_1) - \partial(d_1) = 0$
3. $d(\Delta_1) + \delta(d_1) = 0$
4. $d(\mu_1^3) + \partial(\mu_1) = 0$
5. $d(\Delta_1^3) - \delta(\Delta_1) = 0$
6. $\partial(\mu_1^3) = 0$
7. $\delta(\Delta_1^3) = 0$
8. $d(\omega_1) + \partial(\Delta_1) + \delta(\mu_1) = 0$
9. $\partial(\Delta_1^3) + \delta(\omega_1) = 0$
10. $\partial(\omega_1) - \delta(\mu_1^3) = 0$.

Requiring (H, d_t, μ_t, μ_t^3) and $(H, d_t, \Delta_t, \Delta_t^3)$ to be simple $A(3)$ -(co)algebras tells us that relations (1) - (7) are linearizations of Stasheff's strict $A(4)$ -(co)algebra relations, and relation (8) is the linearization of the Hopf relation relaxed up to homotopy. Since μ_t , ω_t and Δ_t have no terms of order zero, relations (9) and (10) are the respective linearizations of new relations (9) and (10) below. Thus we obtain the following structure relations in H_t :

1. $d_t^2 = 0$
2. $d_t \mu_t = \mu_t (d_t \otimes 1 + 1 \otimes d_t)$
3. $\Delta_t d_t = (d_t \otimes 1 + 1 \otimes d_t) \Delta_t$
4. $d_t \mu_t^3 + \mu_t^3 (d_t \otimes 1 \otimes 1 + 1 \otimes d_t \otimes 1 + 1 \otimes 1 \otimes d_t) = \mu_t (1 \otimes \mu_t) - \mu_t (\mu_t \otimes 1)$
5. $(d_t \otimes 1 \otimes 1 + 1 \otimes d_t \otimes 1 + 1 \otimes 1 \otimes d_t) \Delta_t^3 + \Delta_t^3 d_t = (\Delta_t \otimes 1) \Delta_t - (1 \otimes \Delta_t) \Delta_t$
6. $\mu_t^3 (\mu_t \otimes 1 \otimes 1 - 1 \otimes \mu_t \otimes 1 + 1 \otimes 1 \otimes \mu_t) = \mu_t (\mu_t^3 \otimes 1 + 1 \otimes \mu_t^3)$
7. $(\Delta_t \otimes 1 \otimes 1 - 1 \otimes \Delta_t \otimes 1 + 1 \otimes 1 \otimes \Delta_t) \Delta_t^3 = (\Delta_t^3 \otimes 1 + 1 \otimes \Delta_t^3) \Delta_t$
8. $(d_t \otimes 1 + 1 \otimes d_t) \omega_t + \omega_t (d_t \otimes 1 + 1 \otimes d_t) = \Delta_t \mu_t - (\mu_t \otimes \mu_t) \sigma_{2,2} (\Delta_t \otimes \Delta_t)$
9. $(\mu_t \otimes \omega_t) \sigma_{2,2} (\Delta_t \otimes \Delta_t) - (\Delta_t \otimes 1 - 1 \otimes \Delta_t) \omega_t - (\omega_t \otimes \mu_t) \sigma_{2,2} (\Delta_t \otimes \Delta_t)$
 $= \Delta_t^3 \mu_t - \mu_t^{\otimes 3} \sigma_{3,2} [(\Delta_t \otimes 1) \Delta_t \otimes \Delta_t^3 + (\Delta_t^3 \otimes (1 \otimes \Delta_t) \Delta_t)]$
10. $(\mu_t \otimes \mu_t) \sigma_{2,2} (\Delta_t \otimes \omega_t) - \omega_t (\mu_t \otimes 1 - 1 \otimes \mu_t) - (\mu_t \otimes \mu_t) \sigma_{2,2} (\omega_t \otimes \Delta_t)$
 $= [\mu_t (\mu_t \otimes 1) \otimes \mu_t^3 + \mu_t^3 \otimes \mu_t (1 \otimes \mu_t)] \sigma_{2,3} \Delta_t^{\otimes 3} - \Delta_t \mu_t^3.$

By dropping the formal deformation parameter t , we obtain the structure relations in a *quasi-simple* $A(3)$ -bialgebra.

The first *non-operadic* example of a general A_∞ -bialgebra appears here as a quasi-simple $A(3)$ -bialgebra and involves a non-trivial operation $\omega = \omega^{2,2}$. The six additional relations satisfied by A_∞ -bialgebras of this particular form will be verified in the next section.

Example 2. Let H be the primitively generated bialgebra $\Lambda(x, y)$ with $|x| = 1$, $|y| = 2$, trivial differential, and $\omega : H^{\otimes 2} \rightarrow H^{\otimes 2}$ given by

$$\omega(a|b) = \begin{cases} x|y + y|x, & a|b = y|y \\ x|x, & a|b \in \{x|y, y|x\} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(\Delta \otimes 1 - 1 \otimes \Delta)\omega(y|y) = (\Delta \otimes 1 - 1 \otimes \Delta)(x|y + y|x) = 1|x|y + 1|y|x - x|y|1 - y|x|1 = (\mu \otimes \omega - \omega \otimes \mu)(1|1|y|y + y|1|1|y + 1|y|y|1 + y|y|1|1) = (\mu \otimes \omega - \omega \otimes \mu)\sigma_{2,2}(\Delta \otimes \Delta)(y|y)$; and similar calculations show agreement on $x|y$ and $y|x$ and verifies relation (9). To verify relation (10), note that $\omega(\mu \otimes 1 - 1 \otimes \mu)$ and $(\mu \otimes \mu)\sigma_{2,2}(\Delta \otimes \omega - \omega \otimes \Delta)$ are supported on the subspace spanned by

$$B = \{1|y|y, y|y|1, 1|x|y, x|y|1, 1|y|x, y|x|1\},$$

and it is easy to check agreement on B . Finally, note that (H, μ, Δ, ω) can be realized as the linear deformation $(H[[t]], \mu, \Delta, t\omega)|_{t=1}$.

5. A_∞ -BIALGEBRAS IN PERSPECTIVE

Although G-S deformation cohomology motivates the notion of an A_∞ -bialgebra, G-S deformations of DGBs are less constrained than A_∞ -bialgebras and consequently fall short of the mark. To indicate of the extent of this shortfall, let us identify those structure relations that fail to appear via deformation cohomology but must be verified to assert that Example 2 is an A_∞ -bialgebra.

As mentioned above, structure relations in a general A_∞ -bialgebra arise from the homogeneous components of a square-zero differential on some universal complex. So to begin, let us construct the particular complex that determines the structure relations in an A_∞ -bialgebra of the form $(H, d, \mu, \Delta, \omega^{2,2})$. Given arbitrary maps $d = \omega^{1,1}$, $\mu = \omega^{1,2}$, $\Delta = \omega^{2,1}$, and $\omega^{2,2}$ with $\omega^{j,i} \in \text{Hom}^{3-i-j}(H^{\otimes i}, H^{\otimes j})$, consider $\omega = \sum \omega^{j,i}$. Freely extend

- d as a linear map $(H^{\otimes p})^{\otimes q} \rightarrow (H^{\otimes p})^{\otimes q}$ for each $p, q \geq 1$,
- $d + \Delta$ as a derivation of $T^a H$,
- $d + \mu$ as a coderivation of $T^c H$,
- $\Delta + \omega^{2,2}$ as an algebra map $T^a H \rightarrow T^a(H^{\otimes 2})$, and
- $\mu + \omega^{2,2}$ as a coalgebra map $T^c(H^{\otimes 2}) \rightarrow T^c H$.

The *biderivative* $\overline{\omega}$ is the sum of these free extensions.

Note that in this restricted setting, relation (10) in Definition 2 reduces to

$$(\mu \otimes \mu) \odot (\Delta \otimes \omega^{2,2} - \omega^{2,2} \otimes \Delta) = \omega^{2,2} \odot (\mu \otimes 1 - 1 \otimes \mu).$$

Factors $\mu \otimes 1$ and $1 \otimes \mu$ are components of $\overline{d} + \mu$; factors $\Delta \otimes \omega^{2,2}$ and $\omega^{2,2} \otimes \Delta$ are components of $\overline{\Delta} + \omega^{2,2}$; and the factor $\mu \otimes \mu$ is a component of $\overline{\mu} + \omega^{2,2}$.

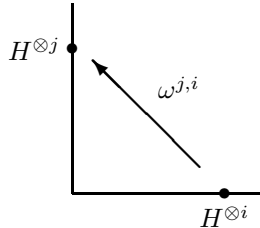


Figure 4. The initial map $\omega^{j,i}$.

To picture this, identify the isomorphic modules $(H^{\otimes p})^{\otimes q} \approx (H^{\otimes q})^{\otimes p}$ with the point $(p, q) \in \mathbb{N}^2$ and picture the initial map $\omega^{j,i} : H^{\otimes i} \rightarrow H^{\otimes j}$ as a “transgressive”

arrow from $(i, 1)$ to $(1, j)$ (see Figure 4). Components of the various free extensions above are pictured as arrows that initiate or terminate on the axes. For example, the vertical arrow $\Delta \otimes \Delta$, the short left-leaning arrow $\Delta \otimes \omega^{2,2} - \omega^{2,2} \otimes \Delta$ and the long left-leaning arrow $\omega^{2,2} \otimes \omega^{2,2}$ in Figure 5 represent components of $\overline{\Delta + \omega^{2,2}}$. Since we are only interested in transgressive quadratic \odot -compositions, it is sufficient to consider the components of $\overline{\omega}$ pictured in Figure 5. Quadratic compositions along the x -axis correspond to relations (1), (2), (4) and (6) in Definition 2; those in the square with its diagonal correspond to relation (8); those in the vertical parallelogram correspond to relation (9); and those in the horizontal parallelogram correspond to relation (10).

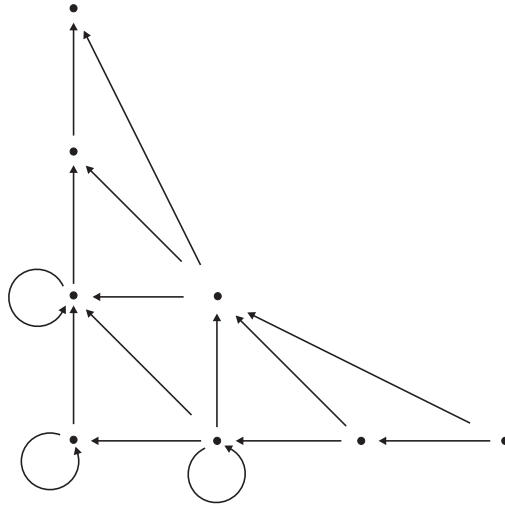


Figure 5. Components of $\overline{\omega}$ when $\omega = d + \mu + \Delta + \omega^{2,2}$.

The following six additional relations are not detected by deformation cohomology because the differentials only detect the interactions between ω and (deformations of) d , μ , and Δ induced by the underlying bi(co)module structure:

11. $(\mu \otimes \omega - \omega \otimes \mu) \sigma_{2,2} (\Delta \otimes \omega - \omega \otimes \Delta) = 0;$
12. $(\mu \otimes \mu) \sigma_{2,2} (\omega \otimes \omega) = 0;$
13. $(\omega \otimes \omega) \sigma_{2,2} (\Delta \otimes \Delta) = 0;$
14. $(\mu \otimes \omega - \omega \otimes \mu) \sigma_{2,2} (\omega \otimes \omega) = 0;$
15. $(\omega \otimes \omega) \sigma_{2,2} (\Delta \otimes \omega - \omega \otimes \Delta) = 0;$
16. $(\omega \otimes \omega) \sigma_{2,2} (\omega \otimes \omega) = 0.$

Definition 3. Let H be a \mathbf{k} -module together with and a family of maps

$$\{d = \omega^{1,1}, \mu = \omega^{1,2}, \Delta = \omega^{2,1}, \omega^{2,2}\}$$

with $\omega^{j,i} \in \text{Hom}^{3-i-j}(H^{\otimes i}, H^{\otimes j})$, and let $\omega = \sum \omega^{j,i}$. Then $(H, d, \mu, \Delta, \omega^{2,2})$ is an A_∞ -bialgebra if $\overline{\omega} \odot \overline{\omega} = 0$.

Example 3. Continuing Example 2, verification of relations (11) - (16) above is straightforward and follows from the fact that $\sigma_{2,2}(y|x|x|y) = -y|x|x|y$. Thus (H, μ, Δ, ω) is an A_∞ -bialgebra with non-operadic structure.

Let H be a graded module and let $\{\omega^{j,i} : H^{\otimes i} \rightarrow H^{\otimes j}\}_{i,j \geq 1}$ be an arbitrary family of maps. Given a diagonal Δ_P on the permutahedra and the notion of a Δ_P -(co)derivation, one continues the procedure described above to obtain the general biderivative defined in [SU05]. And as above, the general A_∞ -bialgebra structure relations are the homogeneous components of $\overline{\omega} \odot \overline{\omega} = 0$.

For example, consider an A_∞ -bialgebra $(H, \mu, \Delta, \omega^{j,i})$ with exactly one higher order operation $\omega^{j,i}$, $i+j \geq 5$. When constructing $\overline{\omega}$, we extend μ as a coderivation, identify the components of this extension in $\text{Hom}(H^{\otimes i}, H^{\otimes j})$ with the vertices of the permutahedron P_{i+j-2} , and identify $\omega^{j,i}$ with its top dimensional cell. Since μ , Δ and $\omega^{j,i}$ are the only operations in H , all compositions involving these operations have degree 0 or $3-i-j$, and k -faces of P_{i+j-2} in the range $0 < k < i+j-3$ are identified with zero. Thus the extension of $\omega^{j,i}$ as a Δ_P -coderivation only involves the primitive terms of $\Delta(P_{i+j-2})$, and the components of this extension are terms in the expression $\delta(\omega^{j,i})$. Indeed, whenever $\omega^{j,i}$ and its extension are compatible with the underlying DGB structure, the relation $\delta(\omega^{j,i}) = 0$ is satisfied. Dually, we have $\partial(\omega^{j,i}) = 0$ whenever $\omega^{j,i}$ and its extension as a Δ_P -derivation are compatible with the underlying DGB structure. These structure relations can be expressed as commutative diagrams in the integer lattice \mathbb{N}^2 (see Figures 7 and 8 below).

Definition 4. Let $n \geq 3$. A simple Hopf $A(n)$ -algebra is a tuple $(H, d, \mu, \Delta, \mu^n)$ with the following properties:

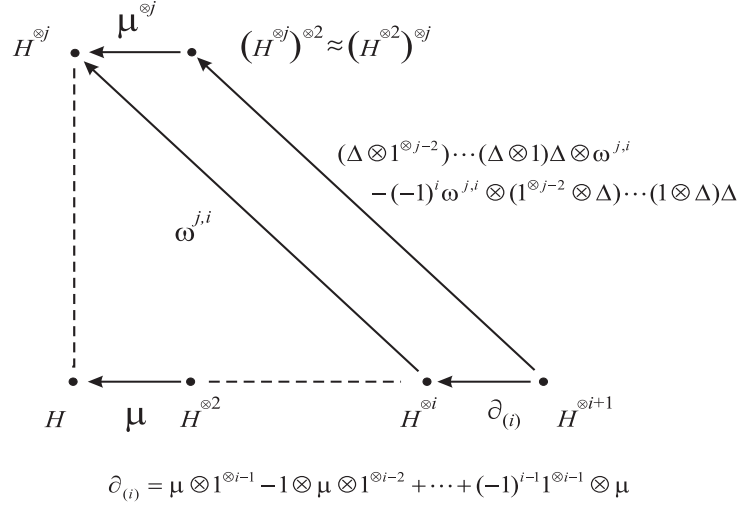
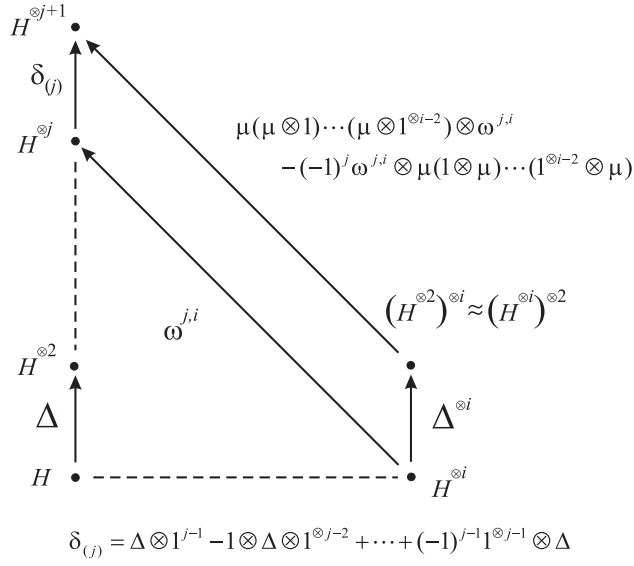
1. (H, d, Δ) is a coassociative DGC;
2. (H, d, μ, μ^n) is an $A(n)$ -algebra; and
3. $\Delta \mu^n = [\mu(\mu \otimes 1) \cdots (\mu \otimes 1^{\otimes n-2}) \otimes \mu^n + \mu^n \otimes \mu(1 \otimes \mu) \cdots (1^{\otimes n-2} \otimes \mu)] \sigma_{2,n} \Delta^{\otimes n}$.

A simple Hopf A_∞ -algebra $(H, d, \mu, \Delta, \mu^n)$ is a simple Hopf $A(n)$ -algebra satisfying the relation in offset (4.1) above. There are the completely dual notions of a simple Hopf $A(n)$ -coalgebra and a simple Hopf A_∞ -coalgebra.

General Hopf A_∞ -(co)algebras were defined by A. Berciano and this author in [BU08]; A_∞ -bialgebras with operations exclusively of the forms $\omega^{j,1}$ and $\omega^{1,i}$, called *special A_∞ -bialgebras*, were considered by this author in [Umb08].

Simple Hopf $A(n)$ -algebras are especially interesting because their structure relations can be controlled by G-S deformation theory. In fact, if $n \geq 3$ and $H_t = (H[[t]], d_t, \mu_t, \Delta_t, \mu_t^n)$ is a deformation, then $\mu_t^n = t\mu_1^n + t^2\mu_2^n + \cdots$ has no term of order zero. Consequently, if $D(\mu_1^n) = 0$, then $t\mu_1^n$ automatically satisfies the required structure relations in a simple Hopf $A(n)$ -algebra and $(H[[t]], d, \mu, \Delta, t\mu_1^n)$ is a *linear* deformation of H_0 as a simple Hopf $A(n)$ -algebra. This proves:

Theorem 1. If (H, d, μ, Δ) is a DGB and $\mu_1^n \in \text{Hom}^{2-n}(H^{\otimes n}, H)$, $n \geq 3$, is a 2-cocycle, then $(H[[t]], d, \mu, \Delta, t\mu_1^n)$ is a linear deformation of H_0 as a simple Hopf $A(n)$ -algebra.


 Figure 7. The structure relation $\partial(\omega^{j,i}) = 0$.

 Figure 8. The structure relation $\delta(\omega^{j,i}) = 0$.

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REFERENCES

- [Bau98] Baues, H.J.: The cobar construction as a Hopf algebra and the Lie differential. *Invent. Math.*, **132**, 467–489 (1998)
- [BU08] Berciano, A., Umble, R.: Some naturally occurring examples of A_∞ -bialgebras. Preprint arXiv:0706.0703.
- [Car55] Cartier, P.: Cohomologie des Coalgebras. In: Séminaire Sophus Lie, Exposé 5, (1955-56)
- [CS04] Chas, M., Sullivan, D.: Closed string operators in topology leading to Lie bialgebras and higher string algebra. The legacy of Niels Henrik Abel, 771–784, Springer, Berlin, (2004)
- [Ger63] Gerstenhaber, M.: The cohomology structure of an associative ring. *Ann. of Math.*, **78**(2) 267–288 (1963)
- [GS92] Gerstenhaber, M., Schack, S.D.: Algebras, bialgebras, quantum groups, and algebraic deformations. In: Contemporary Math. 134. AMS, Providence (1992)
- [God08] Godin, V.: Higher String topology operations. Preprint arXiv:0711.4859.
- [Gug82] Gugenheim, V.K.A.M.: On a perturbation theory for the homology of the loop space. *J. Pure Appl. Algebra*, **25** 197–205 (1982)
- [HKR62] Hochschild, G., Kostant, B., Rosenberg, A.: Differential forms on regular affine varieties. *Trans. AMS*, **102** 383–408 (1962)
- [Kad80] Kadeishvili, T.: On the homology theory of fibre spaces. *Russian Math. Survey*, **35** 131–138 (1980)
- [LM91] Lazarev, A., Movshev, M.: Deformations of Hopf algebras. *Russian Math. Surveys* (translated from Russian) **42** 253–254 (1991)
- [LM96] Lazarev, A., Movshev, M.: On the cohomology and deformations of differential graded algebras. *J. Pure Appl. Algebra*, **106** 141–151 (1996)
- [Lod06] Loday, J.-L.: Generalized bialgebras and triples of operads. Preprint math.QA/0611885.
- [Lod07] ———: The diagonal of the Stasheff polytope. Preprint arXiv:0710.0572.
- [LU92] Lupton, G., Umble, R.: Rational homotopy types with the rational cohomology of stunted complex projective space. *Canadian J. Math.*, **44**(6) 1241–1261 (1992)
- [Mac67] MacLane, S.: Homology. Springer-Verlag, Berlin/New York (1967)
- [Mar06] Markl, M.: A resolution (minimal model) of the PROP for bialgebras. *J. Pure and Appl. Algebra*, **205** 341–374 (2006). Preprint math.AT/0209007.
- [Mar06] ———: Operads and PROPs. Preprint math.AT/0601129.
- [MS06] Markl, M., Shnider, S.: Associahedra, cellular W -construction and products of A_∞ -algebras. *Trans. AMS*, **358**(6) 2353–2372 (2006)
- [MSS02] Markl, M., Shnider, S., Stasheff, J.: Operads in Algebra, Topology and Physics. *Mathematical Surveys and Monographs* **96**, AMS, Providence (2002)
- [McC98] McCleary, J. (ed): Higher Homotopy Structures in Topology and Mathematical Physics. *Contemporary Mathematics* **227**, AMS, Providence (1998)
- [May72] May, J.P.: Geometry of Iterated Loop Spaces. *SLNM* **271**, Springer, Berlin (1972)
- [Pir02] Pirashvili, T.: On the PROP corresponding to bialgebras. *Cah. Topol. Géom. Différ. Catég.*, **43**(3) 221–239 (2002)
- [San96] Saneblidze, S.: The formula determining an A_∞ -coalgebra structure on a free algebra. *Bull. Georgian Acad. Sci.*, **154** 351–352 (1996)
- [San99] ———: On the homotopy classification of spaces by the fixed loop space homology. *Proc. A. Razmadze Math. Inst.*, **119** 155–164 (1999)
- [SU00] Saneblidze, S., Umble, R.: A diagonal on the associahedra. Preprint math.AT/0011065.
- [SU04] Saneblidze, S., Umble, R.: Diagonals on the permutahedra, multiplihedra and associahedra. *J. Homology, Homotopy and Appl.*, **6**(1) 363–411 (2004)
- [SU05] ———: The biderivative and A_∞ -bialgebras. *J. Homology, Homotopy and Appl.*, **7**(2) 161–177 (2005)

- [SU08a] ———: Matrads, matrahedra and A_∞ -bialgebras. J. Homology, Homotopy and Appl., to appear. Preprint math.AT/0508017.
- [SU08b] ———: The category of A_∞ -bialgebras. In preparation.
- [Sho03] Shoikhet, B.: The CROCs, non-commutative deformations, and (co)associative bialgebras. Preprint math.QA/0306143.
- [Sta63] Stasheff, J.: Homotopy associativity of H -spaces I, II. Trans. AMS **108** 275–312 (1963)
- [Sul07] Sullivan, D.: String Topology: Background and Present State. Preprint arXiv:0710.4141.
- [Ton97] Tonks, A.: Relating the associahedron and the permutohedron. In: "Operads: Proceedings of the Renaissance Conferences (Hartford CT / Luminy Fr 1995)" Contemporary Mathematics, **202** 33–36 (1997)
- [Umb89] Umble, R.: Homotopy conditions that determine rational homotopy type. J. Pure and Appl. Algebra, **60** 205–217 (1989)
- [Umb96] ———: The deformation complex for differential graded Hopf algebras. J. Pure Appl. Algebra, **106** 199–222 (1996)
- [Umb08] ———: Structure relations in special A_∞ -bialgebras. J. Mathematical Sciences, to appear. Preprint arXiv:math.AT/0506446.
- [Val04] Vallette, B.: Koszul duality for PROPs. Trans. AMS, **359** 4865–4943 (2007). Preprint arXiv:math/0411542

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